## Turbulent Flows

Stephen B. Pope

Cambridge University Press (2000)

## Solution to Exercise 13.10

Prepared by: Daniel W. Meyer
Date: 15/6/06

In Table 13.2 the transfer function of the sharp spectral filter is given as

$$
\hat{G}(\kappa)=\left\{\begin{array}{l}
0 \text { if }|\kappa| \geq \kappa_{c}  \tag{1}\\
1 \text { if }|\kappa|<\kappa_{c} .
\end{array}\right.
$$

Together with the Kolmogorov spectrum (Eq.(6.239)) we have from Eq.(13.65)

$$
\begin{equation*}
\left\langle k_{r}\right\rangle=\int_{\kappa_{c}}^{\infty} C \varepsilon^{2 / 3} \kappa^{-5 / 3} d \kappa=-\left.\frac{3}{2} C \varepsilon^{2 / 3} \kappa^{-2 / 3}\right|_{\kappa=\kappa_{c}} ^{\infty}=\frac{3}{2} C \varepsilon^{2 / 3} \kappa_{c}^{-2 / 3}, \tag{2}
\end{equation*}
$$

and with the lengthscale $L=k^{3 / 2} / \varepsilon$ the dissipation rate $\varepsilon$ can be eliminated, leading to

$$
\begin{equation*}
\frac{\left\langle k_{r}\right\rangle}{k}=\frac{3}{2} C\left(\kappa_{c} L\right)^{-2 / 3} . \tag{3}
\end{equation*}
$$

This is equivalent to Eq.(13.66). If $80 \%$ of the spectrum is resolved, $20 \%$ are residual motions. Setting Eq.(3) equal to 0.2 and using $C=1.5$ for the universal Kolmogorov constant (Eq.(6.239) below) leads $\kappa_{c} L \approx 38$. With $\ell_{E I}=\frac{0.43}{6} L$ for high-Reynolds-number turbulence and $\kappa_{c}=\pi / \Delta$, the corresponding filter width is

$$
\begin{equation*}
\frac{\Delta}{\ell_{E I}}=\frac{6 \pi}{0.43 \kappa_{c} L} \approx 1.16 . \tag{4}
\end{equation*}
$$

In Table 13.2 the transfer function of the Gaussian filter is given,

$$
\begin{equation*}
\hat{G}(\kappa)=\exp \left(-\frac{\kappa^{2} \Delta^{2}}{24}\right) . \tag{5}
\end{equation*}
$$

Together with the Kolmogorov spectrum one gets from Eq.(13.65)

$$
\left\langle k_{r}\right\rangle=\int_{0}^{\infty}\left[1-\exp \left(-2 \frac{\kappa^{2} \Delta^{2}}{24}\right)\right] C \varepsilon^{2 / 3} \kappa^{-5 / 3} d \kappa
$$

$$
\begin{align*}
& =C \varepsilon^{2 / 3} \int_{0}^{\infty}\left[1-\exp \left(-\frac{\kappa^{2} \Delta^{2}}{12}\right)\right] \kappa^{-5 / 3} d \kappa \\
& =C \varepsilon^{2 / 3} \int_{0}^{\infty}\left(1-e^{-x}\right)\left(\frac{\sqrt{12 x}}{\Delta}\right)^{-5 / 3} \frac{\sqrt{12}}{2 \Delta} \frac{d x}{\sqrt{x}} \\
& =k C 96^{-1 / 3}\left(\frac{\Delta}{L}\right)^{2 / 3} \int_{0}^{\infty}\left(1-e^{-x}\right) x^{-4 / 3} d x . \tag{6}
\end{align*}
$$

On the third line of Eq.(6), $\kappa$ was substituted by $x=\kappa^{2} \Delta^{2} / 12$ and on the fourth line, $\varepsilon$ was eliminated using $L=k^{3 / 2} / \varepsilon$. Dividing Eq.(6) through $k$ leads Eq.(13.69). In Eq.(6), if $\Delta$ is replaced by $\pi / \kappa_{c}\left(\kappa_{c}=\pi / \Delta\right)$ and if the integral is substituted by $I_{0}$ (Eq.(13.70)) we get

$$
\begin{equation*}
\kappa_{c} L=\pi\left(\frac{5}{96^{1 / 3}} C I_{0}\right)^{3 / 2} . \tag{7}
\end{equation*}
$$

Numerical evaluation leads $I_{0} \approx 4.06235$ and together with $C=1.5, \kappa_{c} L \approx$ 54 is resulting. The corresponding filter width is then from Eq.(4) $\Delta / \ell_{E I} \approx$ 0.813 .

This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike License. To view a copy of this license, visit http://creativecommons.org/licenses/by-nc-sa/1.0 or send a letter to Creative Commons, 559 Nathan Abbott Way, Stanford, California 94305, USA.

